On solutions of the standard-model Lagrangian with a Majorana mass term

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ABSTRACT. It is demonstrated that the standard-model Lagrangian with a Majorana mass term for the neutrino admits no non-trivial solution in the presence of charged leptons. Because the standard model is generally believed to describe the gauge interactions of neutrinos correctly, the Majorana mass term must vanish and thus cannot enable neutrino-less double β decay. More generally, neutrinos with standard-model gauge interactions cannot be Majorana fields. Historical reasons why this conclusion has not been drawn earlier are analyzed.

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1 Introduction

1.1 Majorana's lesson and definition of his field

Ettore Majorana presented a crucial insight with eq.(10) of his ultimate publication "Teoria simmetrica dell'elletrone e del positrone"[1]. In a suitable representation of the γ matrices[1] the real part of a spinor alone is a solution to the free Dirac equation. This can be expressed in a representation independent way as: spinor operators "Ψ" that are self-charge conjugate, i.e. for which:

$$\Psi = \Psi^c \quad \text{(Majorana condition)} \tag{1}$$

fulfill the free Dirac equation. Here the superscript "^{c}" symbolizes the operation of "charge conjugation". With our notation[2] charge conjuga-

[1] Nowadays called "Majorana representation".
[2] Appendix 6 contains the conventions I chose.

tion is complex conjugation in the Majorana representation:

$$\Psi = \Psi^c = \Psi^* \quad \text{(Majorana condition and representation)} \qquad (2)$$

The operation takes a slightly different form in other representations (see appendix 6). Self charge-conjugate neutrinos (i.e. neutrino fields that fulfill condition 1) are called "Majorana neutrinos"[3].

1.2 Aim of this manuscript

Majorana taught us that the equation of motion (or the Lagrangian from which it can be derived) is primary. Majorana fields are one special solution of the free Lagrangian. Whether such a solution is still allowed or even required when further mass and/or interaction terms are added to the free Lagrangian is a non-trivial and fundamental question. Surprisingly it has not been comprehensively addressed up to today. In this manuscript I answer it for an important special case: the inclusion of standard-model interactions and a (non-standard) Majorana mass term.

1.3 Structure of this manuscript

The rest of the introduction (subsection 1.4) reviews Majorana mass terms. Section 2 analyzes the solution of a neutrino Lagrangian with standard-model interactions and the addition of a (non standard-model) Majorana mass term. Section 3 identifies historical stumbling blocks in the way of an earlier understanding of the problem addressed here. Section 4 concludes. Appendix 6 lists notational conventions, appendix 7) reviews chiral spinors for the convenience of the reader, appendix 8 proves that Majorana masses require Majorana solutions and appendix 9 reviews the equivalence of free Majorana and Weyl neutrinos.

1.4 Solutions of a free Lagrangian with a Majorana mass term

The following Lagrangian for a free neutrino "requires" a Majorana solution because it contains a "Majorana mass term"[4]:

$$L_\nu^{\mathrm{free}} = i\bar{\nu}\gamma^\mu \frac{\partial}{\partial x^\mu}\nu - [m_{\mathrm{maj}}\bar{\nu}(\nu)^c/2 + H.C.] \qquad (3)$$

[3]In a recent review[2] my eq.(1) appears as eq.(8) and my eq.(2) as a unnumbered eq. a few lines below eq.(8).

[4]Majorana[1] used only Dirac mass terms that allow but do not require Majorana solutions. The Majorana nature of a Lagrangian's mass term is a sufficient but not necessary condition for the existence of a Majorana-field solution. Majorana-mass terms were introduced by Jehle[3] and its relation to Majorana fermions was clarified by Serpe[4].

"'ν" stands for the neutrino-field operator and $m_{\rm maj}$ is the Majorana mass. An equation of motion for a free neutrino with a Majorana mass - derived from eq.(3) according to the Euler-Lagrange equations is:

$$i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\nu - m_{\rm maj}\nu^{c} = 0 \quad \text{(Majorana equation)} \tag{4}$$

The second term contains the charge conjugate of the field that appears in the first term. Such an equation can only be solved in a non trivial way by inserting a self charge-conjugate field[5]. Otherwise the terms cannot cancel for non-vanishing neutrino fields. For a detailed proof see appendix 8. Because particles cannot be turned to their charge conjugate by any Lorentz boost, this conclusion holds for any Lorentz frame[6]. I summarize this in the following

Theorem A

If the Lagrangian contains a finite Majorana-mass term (defined in eq.(3)), then all solutions in all inertial frames are Majorana fields, i.e. they must fulfill eq.(1).

2 Solutions of the Standard Model Lagrangian with a Majorana mass term

My aim is to answer the following question: What are the solutions to the full Lagrangian valid for a physical neutrino with Majorana-mass and interaction terms?

2.1 Form of the total Lagrangian

While it can never be excluded that "new physics" will modify a theory, it is generally believed that SM[5] *interactions* will always at least remain a good approximation to a "final theory" for the conditions prevalent in current experiments[7]. The situation is of course different for the mass term: here it is currently believed that it might be a Majorana mass term. Such a term is qualitatively different from the usual SM Higgs terms that confer mass to all non-neutrino fermions.

[5]That the Majorana mass term determines the Majorana nature of the field seems to be uncontroversial. Bilenky writes[2]:"(The) nature of neutrinos ... is determined by the type of mass term."

[6]In the words of Pauli[6]:" The ordering between particles and antiparticle solutions is Lorentz invariant."

[7]Bilenky[2] assumes a SM interaction term in his review of Majorana neutrinos because it "perfectly describes existing weak interaction data."

The total neutrino Lagrangian with a SM interaction and Majorana mass term is:

$$L_\nu^{\rm tot} = i\bar{\nu}\gamma^\mu \frac{\partial}{\partial x^\mu}\nu + \frac{g}{2\sqrt{2}}\left[W_\mu^+ \bar{\nu}\gamma^\mu(1-\gamma_5)e^- + H.C.\right] - \left[m_{\rm maj}\bar{\nu}(\nu)^c/2 + H.C.\right] \quad (5)$$

Here e^- symbolizes the electron (or muon/tau) field. g is a gauge coupling constant and W^+ is the massive charged weak boson field. For brevity neutral-current terms have been omitted. Their inclusion would not change any conclusions of this manuscript.
The full Majorana equation for the neutrino field ν with charged-current interaction derived from eq.(5) is:

$$i\gamma^\mu \frac{\partial}{\partial x^\mu}\nu + \frac{g}{2\sqrt{2}}\left[W_\mu^+ \gamma^\mu(1-\gamma_5)e^-\right] - m_{\rm maj}\nu^c = 0 \quad (6)$$

2.2 Ultra-relativistic solutions of the SM Lagrangian

In the ultra-relativistic limit $m/E \rightarrow 0$ the mass term is finite but negligible compared to the kinetic term[8] and we set it to 0 here.
At this point I recommend to consult the appendix 7, to review the definition of "left/right handed states" used here. **Throughout this manuscript they denominate the chirality and *not* the helicity eigenstates.**
Factoring out γ^μ in eq.(6) and writing the fields as column vectors in the chiral representation leads to:

$$i\gamma^\mu \left[\frac{\partial}{\partial x^\mu}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} + \frac{g}{2\sqrt{2}}\sqrt{2}W_\mu^+\begin{pmatrix}0\\ e_2^-\end{pmatrix}\right] = 0 \quad (7)$$

One concludes that the derivative of right handed component of the kinetic term ν_1 must be zero likewise:

$$\frac{\partial}{\partial x^\mu}\nu_1 = \frac{\partial}{\partial x^\mu}\nu_R = 0 \quad (8)$$

Let us choose ν=0 at a distant spatial boundary (all observed neutrinos were "produced" by weak interactions). If $\nu_R = 0$ at the boundary and the space-time derivatives are 0 in general ν_R cannot change and remains

[8] A Lorentz frame in which the direction of motion is the same as it would be for a vanishing rest mass is assumed.

0 everywhere. If ν_R=0 any solution must be left-handed everywhere. I summarize this conclusion following theorem:

Theorem B

Any solution to eq.(6) in the ultra-relativistic limit with a non-vanishing electron amplitude is predominantly left-handed, i.e. its chirality= – 1 to good approximation.

2.3 There can be no Majorana neutrinos with SM interactions

Solutions of a Majorana-massive neutrino created by SM interactions (i.e. of Lagrangian eq.(5)) must fulfill both theorem A ("they must be self-charge conjugate"[2]) and B.("they must be chiral"[5]). My novel insight is that they there is no way that they can be both fulfilled at the same time:

1. Theorem B (subsection 2.2) tells us that all physical ultra-relativistic neutrinos physical are left-handed, i.e. they have a definite chirality = –1.

2. Theorem A (subsection 1.4) tells us that the Majorana equation (eq.(1)) must be fulfilled in all inertial frames. Ultra-relativistic Majorana fields must be self-charge conjugate.

3. Charge conjugation flips chirality (see appendix 7). Therefore the Majorana condition can only be fulfilled for fields that have component of left- and right-handed fields that are equal in amplitude. Such fields are no eigenvector of γ_5, i.e. they have no definite chirality. From requirement 2. we conclude that ultrarelativistic Majorana field cannot have a definite chirality.

Requirements 1. and 3. can obviously not be simultaneously fulfilled.

This contradiction forbids any solution for the physical neutrino field (except the trivial solution ν=0). A SM Lagrangian with Majorana-mass term in the presence of electrons allows no solutions. The wide spread folklore[9] that it allows Majorana-neutrino solutions is thereby erroneous.

Even in theories with "new physics" the neutrino's gauge interactions are expected to be described to good approximation by the SM in processes realized in today's laboratory. Necessary conclusions are that physical neutrinos are no Majorana fields to good approximation. Thereby Ma-

[9] I write "folklore" because - as already mentioned below - the question posed in this manuscript was never raised in a precise manner in the literature. Rather the existence of a Majorana solution of the SM Lagrangian with Majorana-mass term is taken for granted without critical discussion.

jorana mass terms cannot enable neutrino-less double β decay with SM interactions.

3 Why was the conclusion of this manuscript not drawn earlier?

The conclusion of subsection 2.3 (“neutrinos are no Majorana fields”) could have been drawn since physicists endorsed the SM gauge i.e. since about 1979 (the year with Nobel prices for discovering the SM)[10]. Why the long delay? Below I offer some explanations.

3.1 Modifying the SM interaction?

It has been suggested that one should postulate a Lagrangian without neutral-current vector terms in order to allow for Majorana solutions[7]. This proposal modifies the SM interactions of the neutrino. If we want to avoid the exclusion of Majorana neutrinos by the argument in subsection 2.3, we must adopt a purely pseudo-vectorial interaction term also for charged currents. However, such a source term does not violate parity, in flagrant contradiction to experience. This rules out this proposal, at least in this simple form.

3.2 The “practical Majorana-Dirac confusion theorem” is incorrect

Perhaps the most important factor eclipsing the realization that SM interactions forbid Majorana fields is the widely accepted validity of an erroneous “practical Majorana-Dirac confusion theorem”[8]. It asserts that in the ultrarelativistic limit there is no phenomenological difference between Dirac and Majorana neutrinos if they interact only with V-A interactions. It is widely accepted as correct, even though - to my knowledge - no complete proof of the theorem was ever claimed. In particular the present manuscript seems to be the first that systematically discusses the effect of SM charged currents on phenomenological equivalence.
In appendix 9 I review a “Majorana-Dirac confusion theorem” theorem for the **free** case (i.e. the case of a **non-interacting** neutrino). **This** theorem is correct. It was proven in a flurry of papers in the spring of

[10] The question: “Could the conclusion have been drawn after the formulation of V-A (in 1958)?” is difficult to answer. Before the formulation and experimental confirmation of the SM, weak interactions were described by a phenomenological nonrenormalizable current-current theory, valid only at distances large compared to $1/M_W$. Probably one could have already guessed at the result of the present manuscript, but for a firm conclusion the formulation of a consistent physical theory - the SM - was necessary.

1957 (e.g. [9, 10]), i.e. before the V-A structure of the weak current was fully understood. Reviewing the historical literature of the late 1950s to early 1960s (e.g. [11]) I suspect that this theorem was erroneously thought to be valid in general (i.e. even if the neutrino has arbitrary interactions) by many. This over-interpretation tilled the soil for the mistaken acceptance of the "practical Majorana-Dirac confusion theorem" in the early 1980s.
In subsection 3.3 I demonstrate explicitly that ultra-relativistic Majorana and Weyl neutrinos have different charge-current interactions in the SM. Thereby the "practical Majorana-Dirac confusion theorem" does not hold with the V-A interactions prescribed by the SM.

3.3 The non equivalence of ultra-relativistic Majorana and Weyl neutrinos in the presence of a V-A interaction term

Formally the Majorana state that is kinematically equivalent to the Weyl neutrino ν_L is:

$$\nu_M(h=-1) = 1/\sqrt{2}(\nu_L + (\nu_L)^c) \tag{9}$$

The first term is a left-handed neutrino amplitdue, that has helicity = –1. The second term is a right-handed **anti**neutrino amplitude, that likewise has helicity = –1 (see appendix 7 for further explanation). The Majorana state kinematically equivalent to the Weyl antineutrino $(\nu_R)^c$ is:

$$\nu_M(h=+1) = 1/\sqrt{2}((\nu_R)^c + \nu_R) \tag{10}$$

Eqs.(9,10) are the **only** possibilities to express Majorana states of definite helicity. The first term of both states is left-handed and the second is of equal amplitude and right handed, i.e. the states conform to requirement 3. for Majorana field in section (11).
One recognizes immediately that standard-model (V-A) interactions do discriminate the states in eqs.(9,10) from the Weyl states: both contain an amplitude that is right handed, therefore sterile and cannot be produced by SM charged currents[11]. The equivalence of Dirac and Majorana neutrinos in the ultra-relativistic limit ceases to hold in the presence of charged-current V-A interactions.

[11] Seen from a different angle: A production of e.g. state eq.(9) together with an electron would not conserve lepton number. But it is well known that SM interactions do, at least for energies accessible in the laboratory[12].

4 Summary

Whether the physical neutrino is described by a Majorana field is determined solely by the Lagrangian it obeys.
1. All solutions of the standard-model interaction with charged leptons, i.e. including all solutions describing charged-current produced neutrinos, must have a definite chirality (left-handed) in the ultra-relativistic limit.
2. By definition Majorana neutrinos are self-charge conjugate (Majorana condition eq.(1)), a property that holds in all inertial frames.
3. Because charge conjugation flips chirality, Majorana neutrinos must have an equal left-handed and right-handed amplitude in all inertial frames, i.e. they do not have a definite chirality because they are no eigen vectors of γ_5.
Sentence 3. - a corollary from the Majorana condition - and the requirements from the SM interactions (sentence 1.) are therefore in contradiction and cannot be fulfilled simultaneously. Because it seems nearly certain that the standard model describes the weak interactions of neutrinos approximately correctly, neutrinos are probably no Majorana fields. The addition a non-standard model Majorana mass term to the standard-model Lagrangian enforces a Majorana solution. Due to the above contradiction the Lagrangian has no non-trivial solutions describing the production of neutrinos then, i.e. such a mass term is in disagreement with observation and cannot enable neutrino-less double β decay.
In principle neutrinos can have Majorana masses. Neutrinos interact with SM interactions. It was my only aim to show that these two statements cannot both apply to physical neutrinos. Because the truth of the latter seems certain, the former must be wrong.

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Silvia Pezzoni informed me that I did not understand the first thing about Majorana neutrinos during a stroll along the Neckar river in 1994. Alvaro de Rujula constructively criticised draft versions of this manuscript (in 2006). Both inputs were of crucial importance to this manuscript. Valeri Dvoeglazov helpfully requested some clarifications. Thanks!

6 Appendix - notational conventions

The notation used is the same as the one used e.g. by Itzykson & Zuber[5], except that a conventional phase factor in the definition of

charge conjugation is chosen as 1 instead of i. It is also the same as the one used by Landau & Lifsits[13] except that they use a definition of γ_5 that is different by a factor −1. All γ matrices in explicit notation in the usual representations can be found in these textbooks.
The components of the metric tensor $g_{\mu\nu}$ are given as g_{00}=+1, g_{11} = g_{22} = g_{33} = −1, all other components are zero. An arbitrary factor in the definition of charge conjugation is chosen such that:

$$\Psi^c = \gamma^2 \Psi^* \tag{11}$$

in the standard representation. Because the transformation matrices from the conventional to the chiral (spinorial) representation are Hermitian, this relation also holds in this representation.
The chirality operator γ_5 is defined as:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

The matrix is given in the chiral representation.

7 Appendix - chirality and helicity reviewed

The material summarized in this appendix is explained in more detail e.g. in sections 2-2-1 and 2-4-3 of Itzykson & Zuber[5].
Left- and right-handed fields are defined as eigen vectors of the operator γ_5. "Chirality" is their respective eigenvalue. A field Ψ_L for which γ_5 $\Psi_L = -\,\nu_L$ (eigenvalue −1) is usually called "left-handed" and is labelled with the subscript "L". One for which γ_5 $\Psi_R = \Psi_R$ (eigenvalue +1) is called "'right handed" and is labelled with the subscript "R". The terms "left- and right-handed" are **only** used in this sense in this manuscript and **not** as designating helicity. In the chiral representation we can then write the spinor as $\Psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $\Psi_L = \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix}$, $\Psi_R = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}$.
The helicity of a spinor is defined as the normalized scalar product between the particle's momentum and its spin vector. Chirality is not the same as helicity. A left-handed particle has chirality=−1 and helicity=−1. A left-handed **anti**particle has chirality=−1 but its helicity=+1 ([5] eq.(2-103) ff.)[12].

[12] There is a nice intuitive explanation of this fact at the end of section 10.12 of Bjorken/Drell[14]

Charge conjugation "flips chirality" i.e. it takes left handed particles into right handed ones and vice versa. E.g. it follows from eq.(11):

$$(\Psi_L)^c = \gamma^2(\Psi_L)^* = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \phi_2^* \end{pmatrix} = \begin{pmatrix} \sigma^2 \phi_2^* \\ 0 \end{pmatrix} = (\Psi^c)_R. \tag{13}$$

8 Appendix - proof that a Majorana mass term requires a Majorana solution

Let us choose the Majorana representation here. Eq.(4) can be rewritten as:

$$\left[\frac{i\gamma^\mu}{m_{\rm maj}} \frac{\partial}{\partial x^\mu}\right] (\nu_{\rm Re} + i\nu_{\rm Im}) - (\nu_{\rm Re} - i\nu_{\rm Im}) = 0 \tag{14}$$

$\nu_{\rm Re}$ and $\nu_{\rm Im}$ are the real and imaginary amplitude of the neutrino field. If the field is complex these (real) amplitudes and their spatial and temporal derivatives must be equal everywhere in space-time. Because all γ matrices are imaginary[5], the pre-factor $\frac{i\gamma^\mu}{m}$ is real. Eq.(14) can therefore be written as:

$$(c-1)\nu_{\rm Re} + i(c+1)\nu_{\rm Im} = 0 \tag{15}$$

where c is a real number. This equation has no solution for equal $\nu_{\rm Re}$ and $\nu_{\rm Im}$. It can only be solved if one of these amplitudes vanishes, i.e. ν is either real or purely imaginaty. A purely real field is a Majorana field by eq.(2). A purely imaginary field is also a Majorana field, because in this case $\nu^c = -\nu$ holds, which differs from eq.(1) only by a phase factor.

9 Appendix - The equivalence of ultra-relativistic free Majorana and Weyl neutrinos

In the ultra-relativistic limit m/E $\rightarrow$ 0 **free** Dirac[13] and Majorana neutrinos are phenomenologically equivalent[9]. Dirac neutrinos (with lepton number +1) have helicity h=–1, Dirac antineutrinos (with lepton number –1) have helicity h=+1. Helicity is conserved in the ultra-relativistic limit, and it can therefore in principle assume the role of lepton number for Majorana neutrinos.

However, the validity of this equivalence cannot be simply taken for granted if the field interacts. As an obvious counter example: a purely neutral-current vectorial interaction (like "electrical charge") is possible

[13] Massless Dirac fields were first studied and recognized as 2-state systems by Weyl[15]. "Weyl neutrinos" do not obey the Majorana condition (eq.(1)).

for Dirac but not for Majorana fields, thus discriminating them, even for $m/E \rightarrow 0$[14].

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[14] Scalar (S), pseudo-scalar (iP) and pseudo-vector (A) terms do not change under charge conjugation. A phenomenological Dirac-Majorana equivalence can only hold for fields with these neutral-current interactions.